erator operated by gas injection and used for the introduction of plasmoids into magnetic traps.

References

[1] Pearson, J., Cavalconte, C., Guman, W., and Granet, I., "Design, fabrication and test of a pulsed plasma engine for space applications," *Proceedings, Engineering Aspects of MHD*, edited by C. Manual and N. W. Mather (Columbia University Press, New York, 1962), pp. 81–96.

[2] McIlroy, W. and Siegel, B., "Analysis of factors influencing plasma engine performance," American Society of Mechanical Engineers Paper 63-AHGT-62 (1963).

[3] Mostov, P., Neuringer, J., and Rigney, D., "Electromagnetic acceleration of a plasma slug," Phys. Fluids **4**, 1097–1104 (1961).

[4] Loeb, L. B., *Fundamental Processes of Electrical Discharges in Gases* (John Wiley & Sons, Inc., New York, 1959).

[5] Brown, S. C., *Basic Data of Plasma Physics* (John Wiley & Sons, Inc., New York, 1959).

[6] Maier, F., "Glow discharge in high velocity gas streams," Z. Physik **93**, 65–85 (1934).

[7] Fucks, W., "Dependence of breakdown on dark current and frequency," Appl Sci. Res. **B5**, 109–23 (1955).

[8] Levy, M. E., "Discharge gauge for transient density measurements," Rev. Sci. Instr. **35**, 1232–1233 (September 1964).

[9] Denichev, V. F. and Matyukhin, V. D., Akad. Nauk SSSR, Doklady, **150**, 279–282 (May 1963).

Cross-Flow Effects in the Plane-Wall Jet

L. G. Napolitano* and F. Manzo†
Aerodynamics Institute, University of Naples, Italy

THE similar plane laminar wall jet was first considered in Ref. 1. A more general treatment was given by Glauert[2] who dealt, in an unified manner, with both the plane and axysymmetric cases and gave closed-form solutions. The object of this note is to discuss two instances of the laminar wall jet with cross flow. Specifically, we shall consider the similar flow fields due to 1) a two-dimensional jet impinging diagonally (i.e., nonorthogonally) on an infinite plane, and 2) a jet flowing over both sides of a semi-infinite plate that moves in its own plane with a constant velocity (normal to the jet). Apart from the possible practical interest of these cases, the solutions herein derived are mainly offered as additional simple instances of similar three-dimensional laminar flow fields.

According to the remarks made in Ref. 3, similar solutions for the subject flow fields, if existing at all, must be particular solutions of the ordinary differential equations found in Refs. 4 and 5. Therefore, rather than duplicating the usual procedure connected with the "search for similar solutions" we start directly from these ordinary differential equations.

If (x, y, z) is a cartesian reference frame, z the coordinate normal to the body and (u, v, w) the corresponding velocity components, a particular class of three-dimensional similar flow fields is defined by[4,5]:

$$u = Ux^nF'(\eta) \qquad v = Vx^mG'(\eta)$$

$$w = -\frac{U^{1/2}}{K}x^{(n-1)/2}\left[\frac{n+1}{2}F(\eta) + \frac{n-1}{2}\eta F'(\eta)\right]$$

$$\eta = Kz(Ux^{n-1})^{1/2} \tag{1}$$

where U, V, K, n and m are arbitrary constants, η is the similarity variable, primes denote differentiation with respect to η, and the functions F and G satisfy the equations[4,5]:

$$F''' + FF'' - [2n/(n+1)]F'^2 = 0 \tag{2}$$

$$G''' + FG'' - [2m/(n+1)F'G' = 0 \tag{3}$$

where we have already taken into account that for a wall-jet problem the velocity must vanish[2] far away from the wall (i.e., $F'(\infty) = G'(\infty) = 0$). Since the "independence principle" holds, we can consider Eqs. (2) and (3) separately. We will say that a similar solution for the aforementioned flow fields is "found" if the pertinent physical boundary conditions, when formulated in terms of the functions F and G, are consistent with Eqs. (1) and lead to a solution of Eqs. (2) and (3). In both cases treated herein, the wall is impermeable: hence $u(0) = w(0) = 0$ and, from Eqs. (1), $F(0) = F'(0) = 0$ so that the boundary conditions for Eq. (2) are homogeneous. Glauert[2] has shown that a nontrivial solution without reverse-flow exists only for $n = (-\frac{1}{2})$ and is given by

$$F = h^2 \qquad F' = \tfrac{2}{3}h(1-h^3)$$

$$\eta = \ln\left[\frac{(1+h+h^2)^{1/2}}{1-h}\right] + (3)^{1/2}\, tn^{-1}\left[\frac{h(3)^{1/2}}{2+h}\right] \tag{4}$$

Substitution of these relations into Eq. (3) yields, with $G'(\eta) = H(h)$

$$(d^2H/dh^2) - [24mh/(1-h^3)]H = 0 \tag{5}$$

Consider first the case of a plane jet impinging diagonally on an infinite plane (x being the distance from the axis of the jet). Then $v(0) = v(\infty) = 0$ or, from Eqs. (1) $G'(0) = G'(\infty) = 0$, so that the boundary conditions for Eq. (5) are $H(0) = H(1) = 0$. Nontrivial solutions can be obtained only for well-defined values (eigenvalues) of m. It can be shown that the first few eigenvalues are

$$m_1 = -\tfrac{1}{2} \qquad m_2 = -\tfrac{7}{4} \qquad m_3 = -\tfrac{15}{4}$$

The solution for the first eigenvalue ($m_1 = -\frac{1}{2}$) is rather obviously $H = Ch(1-h^3)$ where C is an arbitrary constant; i.e., G' proportional to F'. Thus the cross-velocity profile is proportional to the main velocity profile. It can be easily argued that the proportionality factor must be equal to the corresponding proportionality factor between the jet velocity components normal and tangential to the wall. The solutions corresponding to the other eigenvalues exhibit a "reverse-flow" behavior. Thus, for instance, the solution for $m = -\frac{7}{4}$ is given by

$$G_1'(\eta) = H(h) = h(1-h^3)(1-\tfrac{5}{2}h^3) \tag{6}$$

as it can be easily checked by direct substitution. This solution is plotted in Fig. 1 together with the solution $G' = h(1-h^3)$ (Glauert profile). At this stage there does not seem to be any support for the actual physical occurrence of these reverse cross-flows, so that, until further evidence, solutions such as that given by Eq. (6) have only a mathematical sig-

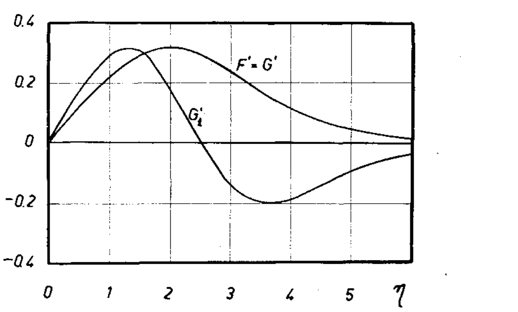

Fig. 1 Cross-velocity profiles for nonorthgonal plane-wall jet corresponding to the first and second eigenvalue.

Received April 26, 1965. This work has been sponsored by the Comitato Nazionale delle Ricerche.

* Director, Institute of Aerodynamics. Member AIAA.

† Research Assistant, Institute of Aerodynamics.

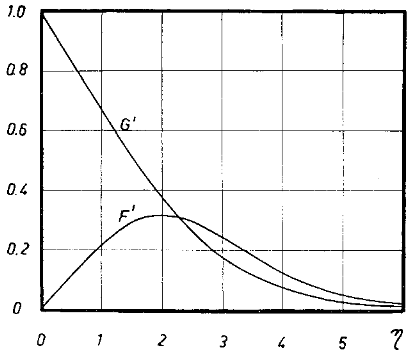

Fig. 2 Velocity profiles for plane-wall jet over a sliding plate.

nificance. [A similar situation arises for the two-dimensional eigenvalue problem (Glauert problem): however, in this case the reverse-flow solutions mentioned by Glauert have not been determined yet].

Consider now the case of a plane jet flowing (in the x direction) over both sides of a semi-infinite plate that moves with a constant velocity (in the y direction). In this case the pertinent boundary conditions are $v(0) = V$; $v(\infty) = 0$ so that, as Eq. (1) shows, $m = 0$ and $G'(0) = H(0) = 1$; $G(\infty)' = H(1) = 0$. The corresponding solutions for G' and G are

$$G' = 1 - h \qquad (3)^{1/2}G = 6\ tn^{-1}[(1 + 2h)/(3)^{1/2}] - \pi$$

The function $G'(\eta) = v/V$ and G are shown in Figs. 2 and 3 where, for comparison, the functions F' and F are also reported. These figures are self explanatory.

The characteristics of the dissipative flow are given by

$$u = (5M/2\nu x)^{1/2}F'(\eta) \qquad v = VG'(\eta) = V(1 - h)$$

$$\eta = \left(\frac{5M}{32\nu^3x^3}\right)^{1/4} z = \ln\left[\frac{(1 + h + h^2)^{1/2}}{1 - h}\right] + (3)^{1/2}\ tn^{-1}\left[\frac{(3)^{1/2}h}{2 + h}\right]$$

$$\frac{\tau_{xz}}{\rho} = \frac{1}{9}\left(\frac{125M^3}{8\nu x^5}\right)^{1/4} \qquad \frac{\tau_{yz}}{\rho} = -\frac{V}{3}\left(\frac{5M\nu}{32x^3}\right)^{1/4}$$

$$w(\infty) = -\frac{1}{4}\left(\frac{40M\nu}{x^3}\right)^{1/4}$$

where ρ is the density, ν the kinematic viscosity, τ_{xz} and τ_{yz} the shear stresses at the wall, and M is a constant related to the "exterior moment flux"[2] of the jet. The expressions for u, η, and τ_{xz} are those given by Glauert.[2] The motion of the plate does not have any influence on the flux of mass ρw entrained into the dissipative region.

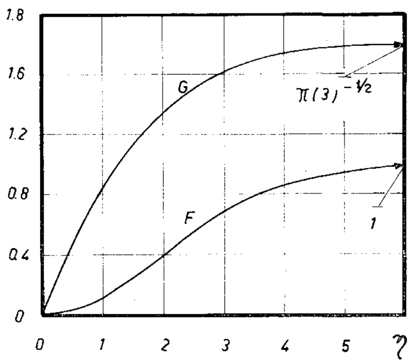

Fig. 3 Functions G and F for plane-wall jet over a sliding plate.

The equation for the limiting streamlines is given by

$$y - y_0 = \frac{2}{3} V \left(\frac{2\nu}{5M}\right)^{1/2}\left[\frac{G'(0) - 1}{F'(0)}\right] x^{3/2} = -V\left(\frac{2\nu}{5M}\right)^{1/2} x^{3/2}$$

with respect to a reference frame fixed on the plate, the "absolute" limiting streamlines being, obviously, the lines $y =$ const.

References

[1] Tetervin, N., "Laminar flow of a slightly viscous incompressible fluid that issues from a slit and passes over a flat plate," NACA TN 1644 (June 1948).

[2] Glauert, M. B., "The wall-jet," J. Fluid Mech. 1, 625–643 (1956).

[3] Napolitano, L. G. and Manzo, F., "On some new classes of similar solutions for three dimensional incompressible boundary layers," AGARD Spring Specialists' Meeting of the Fluid-Dynamics Panel, Naples, Italy, AGARDograph 97 (May 1965).

[4] Geis, T., "Ahnliche dreidimensionale Grenzschiten," J. Rational Mech. Anal. 5, 643–686 (1956).

[5] Herzig, H. Z. and Hansen, A. G., "Similarity with respect to stationary polar coordinates for small angle variation," NACA TN 3890 (1957).

Similar Flow Boundary Layer on Bodies in the Presence of Shear Flow

LEROY DEVAN*
University of California, Los Angeles, Calif.

INCOMPRESSIBLE laminar flow over two-dimensional symmetric bodies described by $y_b = \pm x_b{}^m$ is considered. y and x are dimensionless coordinates (b refers to the body). The dimensionless velocity far upstream ($x = -\infty$) is given by

$$\partial\psi/\partial y = 1 + \Omega\,|y| \qquad m < 1 \tag{1}$$

Equation (1) describes a symmetric shear flow where Ω is a dimensionless vorticity $\Omega = \omega_0 a/U_0$, ψ is a dimensionless outer-stream function, U_0 is the reference velocity, a is a reference length, and ω_0 is the uniform vorticity. The velocity component tangent to the body surface is given by the superposition of two contributions. Uniform flow contributes a velocity that is zero at the stagnation point and approaches 1 asymptotically far downstream. The contribution of the vorticity term is given by a solution of Poisson's equation with source term equal to $\Omega sgn(y)$. A technique for solution of this problem is given in Ref. 1 and is applied in Ref. 2:

$$U_v = \frac{2\xi}{\pi}\frac{d\xi}{ds}\int_0^\infty \frac{(d\psi_b/d\xi^*)d\xi^*}{(\xi^*)^2 - \xi^2} + \Omega|y_b|\cos\theta_b(s) \tag{2}$$

The rotational contribution is given by $\Omega\,|y_b|\cos\theta_b$; θ_b is the angle the tangent to the body surface makes with the positive x axis. s is the dimensionless arc length measured from a stagnation point. The Cauchy principal integral represents the potential solution for the upper half plane. $\psi_b = (-\Omega|y_b|y_b/2)$ represents the boundary value for a Dirichlet problem. ξ is the real axis in the Z plane, and $Z = f(x + iy) = f(z)$ maps the body to the ξ axis. The complex potential $\bar{Z} = \bar{\xi} + i\bar{\eta} = g(z)$ for the uniform flow problem will map the body to a positive real axis. Then $Z^2 = \bar{Z}$ will complete the mapping of the body surface to the real axis. Advantage is taken of the fact that $d\psi_b/d\xi^*$ is even in ξ^*. The

Received February 16, 1965; revision received May 5, 1965. This research was supported by NASA Contract No. NsG 237-62.

* Post Doctoral Scholar, Department of Engineering; now Assistant Professor, Department of Mechanics and Aerospace Engineering, University of Kansas.